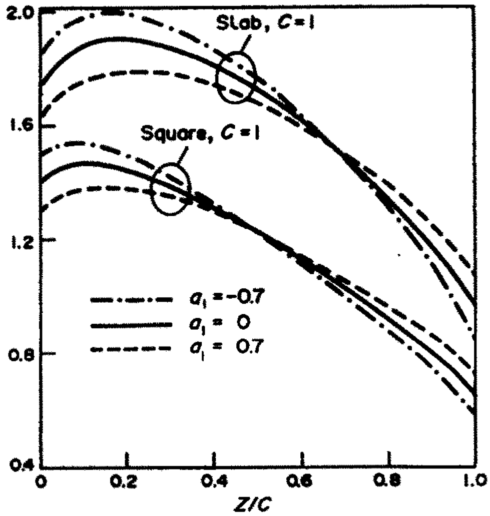

FIG. 2. The total radiation intensity at $y = 0$, $J(0, z)$, for a slab and a square.

Figure 1 shows that the outward flux decreases as the location considered moves to the corner. The corner effects are because part of the radiation is scattered out of the side surface before reaching the bottom. Backward scattering, say $a_1 = -0.7$, always generates the largest outward flux at the bottom, $Q_z^-(y, 0)$, while forward scattering, say $a_1 = 0.7$, generates the smallest, as shown in Fig. 1. This is because radiation originating at a point in the medium or at a boundary is more easily reflected back into the surroundings due to a relatively larger backward scattering. Besides, the incident radiation penetrating a medium decreases with optical size. This tendency is the same as that found in isotropic scattering [5] (see also Table 1).

Figure 2 shows that (i) the total intensity at the locations around the bottom is increased by backward scattering, but is decreased by forward scattering, (ii) the maximum of the total intensity locates about 10% of the optical thickness above the bottom, and (iii) the total intensity in a square medium is less than that in a slab with the same optical thickness.

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A linear stability analysis of a mixed convection plume

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INTRODUCTION

ANALYSES of laminar mixed convection from a horizontal line source of heat have been reported in a number of recent studies. These include the earliest by Wood [1], followed by those of Wesseling [2], Afzal [3] and Krishnamurthy and Gebhart [4]. All these studies were primarily concerned with the predictions of velocity and temperature fields.

In this paper, the stability of such flows to small disturbances is investigated in terms of the linear stability theory. The buoyancy force and the free stream flow are taken to be in the same direction. The region sufficiently downstream of the source is considered, where buoyancy effects dominate. This flow configuration, is usually termed aiding mixed convection.

The effect of the free stream is considered as a perturbation in the far-field boundary condition on the tangential velocity component of the natural convection plume. This perturbation is termed the mixed convection effect and is characterized by the parameter ε_M. Also taken into account is the first-order correction to the 'classical' boundary layer solution to the natural convection plume. This correction results from the interaction of the plume with the irrotational flow outside the boundary layer. This perturbation is termed the higher-order effect and is characterized by ε_H. The base flow is taken to be the classical natural convection plume perturbed by ε_M and ε_H. The stability analysis is then performed by expanding the disturbance field too, in terms of these two perturbation parameters. These two perturbation parameters have been so chosen that at zero order, the governing equations reduce to that of the laminar natural convection plume. Computed results are presented and discussed for $Pr = 0.7$.

ANALYSIS

The mixed convection flow arising from an infinitely long horizontal line source of heat is considered as a two-dimensional steady flow. With the usual Boussinesq approxi-

NOMENCLATURE

A	defined in equation (28)
c	Ω/α
F_i	terms in the expansion for $\bar{\psi}$ in equation (6), $i = 1, 3$
g	acceleration due to gravity
G	$(Gr_x)^{1/5}$
Gr_x	$g\beta Q_0 x^3/k\nu^2$
H_i	terms in the expansion for $(\bar{T}-T_\infty)$ in equation (7), $i = 1, 3$
k	thermal conductivity
Pr	Prandtl number
$Q(x)$	local value of thermal convected energy
Q_0	thermal input per unit length of the line source
$\bar{R}$	parameter defined in equation (26)
R_i	defined in equations (21) and ref. [7]
Re_x	Reynolds number, $U_\infty x/\nu$
S	defined in equation (8)
S_i	defined in equation (14)
$\bar{T}$	base flow temperature
$\tilde{T}$	temperature of the disturbance
ΔT	characteristic temperature difference between the centerline and the edge of the plume, $Q_0\delta/kx$
$\bar{u}$	base flow velocity component in the x-direction
$\tilde{u}$	disturbance velocity component in the x-direction
U	characteristic plume velocity in the x-direction, $\nu G^2/x$
$\bar{v}$	base flow velocity component in the y-direction
$\tilde{v}$	disturbance velocity component in the y-direction
W	defined in equation (25)
x	vertical coordinate along the plume centerline
y	coordinate normal to x.

Greek symbols

α	$\delta\, \mathrm{d}\Lambda/\mathrm{d}x$
β	coefficient of thermal expansion
γ	$\Omega(\mathrm{d}\Omega/\mathrm{d}\alpha_3)$
δ	characteristic thickness of the plume, x/G
ε_H	$1/G$
ε_M	Re_x/G^2
$\bar{\zeta}$	$\partial\bar{v}/\partial x-\partial\bar{u}/\partial y$
$\tilde{\zeta}$	$\partial\tilde{v}/\partial x-\partial\tilde{u}/\partial y$
η	y/δ
Λ	defined in equations (8) and (9)
ν	kinematic viscosity
ϕ	defined in equation (9)
ϕ_i	terms defined in equation (15)
τ	time
$\bar{\psi}$	base flow stream function
$\tilde{\psi}$	stream function for disturbance field
ω	frequency of disturbance
Ω	$\omega\delta/U$.

mations, neglecting viscous dissipation and pressure terms in the energy equation, the full two-dimensional governing equations take the form

$$\psi_y \frac{\partial}{\partial x}(\nabla^2\psi)-\psi_x\frac{\partial}{\partial x}(\nabla^2\psi)-\nu\nabla^4\psi-g\beta\frac{\partial T}{\partial y}=0 \quad (1)$$

$$\psi_y\frac{\partial T}{\partial x}-\psi_x\frac{\partial T}{\partial y}=\frac{\nu}{Pr}(T_{xx}+T_{yy}) \quad (2)$$

where the stream function ψ has been so defined that

$$u=\psi_y \quad \text{and} \quad v=-\psi_x.$$

Boundary conditions are

$$y=0, \quad \psi=\psi_{yy}=T_y=0; \quad \text{for all } x \quad (3)$$

$$y\to\infty, \quad \psi_y\to U_\infty, \quad T\to T_\infty; \quad \text{for all } x. \quad (4)$$

Also for $x>0$, the convected energy is

$$Q(x)=\int_{-\infty}^{\infty}\rho c_p\psi_y(T-T_\infty)\,\mathrm{d}y=Q_0=\text{constant} \quad (5)$$

where Q_0 is the thermal input per unit length of the line source.

In the region $y<O(\delta)$, the base flow can be represented as

$$\bar{\psi}=U\delta(F_1(\eta)+\varepsilon_M F_2(\eta)+\varepsilon_H F_3(\eta)) \quad (6)$$

and

$$\bar{T}-T_\infty=\Delta T(H_1(\eta)+\varepsilon_M H_2(\eta)+E_H H_3(\eta)) \quad (7)$$

where the governing equations and corresponding boundary conditions for F_i and H_i, $i=1$ and 3 are given in ref. [7] and those for $i=2$ are given in the Appendix.

In the usual manner for linear stability analyses, we superimpose on the base flow an arbitrarily small disturbance of the form

$$\tilde{\psi}=U\delta S(\eta)\exp(\mathrm{i}(\Lambda(x)-\omega\tau))+\text{c.c.} \quad (8)$$

$$\tilde{T}=\Delta T\phi(\eta)\exp(\mathrm{i}(\Lambda(x)-\omega\tau))+\text{c.c.} \quad (9)$$

where 'c.c.' denotes complex conjugate and S, ϕ and Λ are complex and ω is taken to be real. Also, $\tilde{u}=\tilde{\psi}_y$ and $\tilde{v}=-\tilde{\psi}_x$.

Each flow variable is represented by the sum of its base flow component and the disturbance component. Then by subtracting the base flow equations from the complete two-dimensional, time-dependent governing equations and combining the x- and y-momentum equations to eliminate pressure terms, the vorticity and energy equations for the disturbance components are obtained. The linearized forms of these equations are given as

$$\frac{\partial\tilde{\zeta}}{\partial\tau}+\tilde{u}\frac{\partial\bar{\zeta}}{\partial x}+\bar{u}\frac{\partial\tilde{\zeta}}{\partial x}+\tilde{v}\frac{\partial\bar{\zeta}}{\partial y}+\bar{v}\frac{\partial\tilde{\zeta}}{\partial y}=\nu\nabla^2\tilde{\zeta}-g\beta\frac{\partial\tilde{T}}{\partial y} \quad (10)$$

$$\frac{\partial\tilde{T}}{\partial\tau}+\tilde{u}\frac{\partial\bar{T}}{\partial x}+\bar{u}\frac{\partial\tilde{T}}{\partial x}+\tilde{v}\frac{\partial\bar{T}}{\partial y}+\bar{v}\frac{\partial\tilde{T}}{\partial y}=\frac{\nu}{Pr}(\nabla^2\tilde{T}). \quad (11)$$

In stability analyses of natural convection boundary layers, an approach that has been successfully used in the past is to exploit the linearity of the disturbance equations by representing the disturbance field as

$$S=S_1+B_2S_2+B_3S_3 \quad (12)$$

$$\phi=\Phi_1+B_2\Phi_2+B_3\Phi_3 \quad (13)$$

where each (S_j, Φ_j) is an integral of the coupled Orr–Sommerfeld equations, with $j=1$ corresponding to the inviscid limit and $j=2, 3$, being characterized by viscous effects. This very approach has also been successfully used by Carey and Gebhart [5] in analyzing the stability of an aiding mixed convection boundary layer flow adjacent to a vertical uniform-flux surface. However, in boundary-free flows such as plumes, B_2 and B_3 have to be identically zero as pointed out by Lin [6] and discussed by Hieber and Nash [7]. A more appropriate method is to expand the disturbance field in terms of the perturbation parameters as in ref. [7]. Thus

$$S = S_1(\eta)+\varepsilon_M S_2(\eta)+\varepsilon_H S_3(\eta) \tag{14}$$

$$\phi = \phi_1(\eta)+\varepsilon_M\phi_2(\eta)+\varepsilon_H\phi_3(\eta) \tag{15}$$

$$\Lambda = \Lambda_1(x)+\varepsilon_M\Lambda_2(x)+\varepsilon_H\Lambda_3(x). \tag{16}$$

Additional quantities that arise are, non-dimensional frequency Ω, complex wave number α and the complex wave speed c, given by

$$\Omega = \delta\omega/U$$

$$\alpha = \delta\frac{d\Lambda}{dx} = \alpha_1+\varepsilon_M\alpha_2+\varepsilon_H\alpha_3 \tag{17}$$

$$c = \Omega/\alpha = c_1+\varepsilon_M c_2+\varepsilon_H c_3. \tag{18}$$

Here, the value of ω will be taken as real.

Substituting the expressions for $\bar{u}$, $\tilde{u}$, $\bar{v}$, $\tilde{v}$, $\bar{T}$, $\tilde{T}$, $\bar{\zeta}$ and $\tilde{\zeta}$ into equations (10) and (11) and ordering the terms in terms of ε_M and ε_H, the following equations result.

At zero order:

$$\mathscr{L}(S_1) \equiv (F_1'\alpha_1-\Omega)(S_1''-\alpha_1^2 S_1)-\alpha_1 F_1''' S_1 = 0 \tag{19}$$

$$\phi_1 = H_1' S_1\alpha_1/(F_1'\alpha_1-\Omega). \tag{20}$$

At $O(\varepsilon_M)$:

$$\mathscr{L}(S_2) = R_1+\alpha_2 R_2 \tag{21}$$

where

$$R_1 = -F_2'\alpha_1(S_1''-\alpha_1^2 S_1)+\alpha_1 S_1 F_2'''$$

and

$$R_2 = 2\alpha_1 S_1(F_1'\alpha_1-\Omega)$$

$$\phi_2 = (\alpha_2(H_1 S_1-F_1\phi_1)+\alpha_1(S_2 H_1'+S_1 H_2' -F_2'\phi_1))/(F_1'\alpha_1-\Omega). \tag{22}$$

The equations at $O(\varepsilon_H)$ are the same as those in ref. [7]. The boundary conditions are that, $S_j(0) = S_j(\infty) = 0$; $j = 1, 3$. The choice of the first boundary condition has been made on the basis of measurements in a natural convection plume reported by Pera and Gebhart [8]. They found this mode of the disturbance to be more unstable than the symmetric mode. Such measurements in mixed convection plumes are not yet available. The equations at zero order and at $O(\varepsilon_H)$ are essentially the same as those in ref. [7]. It is to be noted that the base flow has not been assumed to be parallel. The terms representing the non-parallel nature of the base flow are included in the governing equations for the disturbance at $O(\varepsilon_H)$. Since the homogeneous problems for S_2 and S_3 are the same as those for S_1, it is required that

$$\int_0^\infty (R_1+\alpha_2 R_2)W\,d\eta = 0 \tag{23}$$

and

$$\int_0^\infty (R_3+\alpha_3 R_4)W\,d\eta = 0 \tag{24}$$

where $W(\eta)$ is a non-trivial solution of the adjoint homogeneous problem

$$(F_1'\alpha_1-\Omega)(W''-\alpha_1^2 W)+2F_1''\alpha_1 W = 0 \tag{25}$$

with

$$W'(0) = W(\infty) = 0.$$

From equations (23) and (24), it is easy to see that

$$\alpha_2 = -\int_0^\infty R_1 W\,d\eta \Big/ \int_0^\infty R_2 W\,d\eta$$

and

$$\alpha_3 = -\int_0^\infty R_3 W\,d\eta \Big/ \int_0^\infty R_4 W\,d\eta.$$

The chosen value of Ω, equation (19), is solved to determine S_1 and α_1. Then $W(\eta)$ is determined from equation (25). With α_1 and W known, α_2 can then be determined from equations (23). The procedure for obtaining α_3 is similar and is given in ref. [7].

The two perturbation parameters ε_M and ε_H arise from distinct physical considerations. Yet the two can be related by

$$\varepsilon_M = \bar{R}\varepsilon_H^{1/3} \tag{26}$$

where

$$\bar{R} = U_\infty(\nu^2 k/g\beta Q_0)^{1/3}/\nu.$$

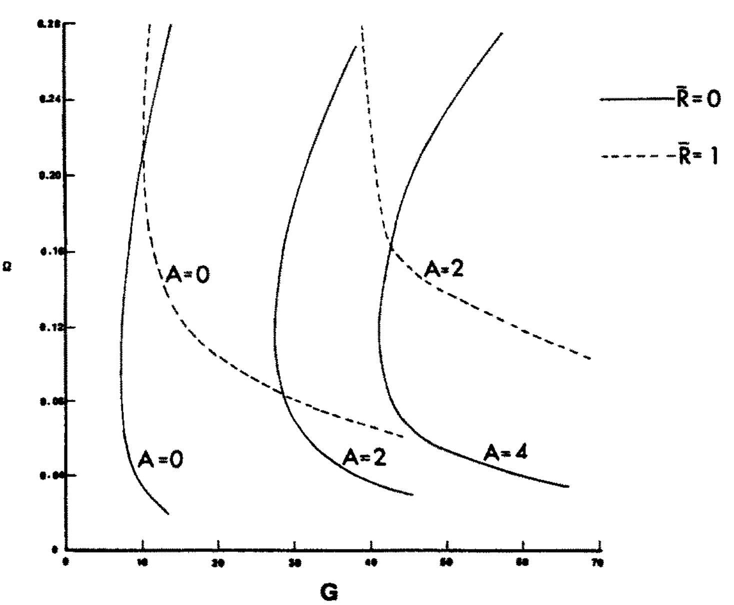

FIG. 1. Contours of constant amplification in natural (——) and mixed convection plumes (– – –).

Clearly $\bar{R}$ is independent of x. If $\bar{R}$ is $O(1)$ or smaller, then the effect of mixed convection on the stability of the flow is inviscid in nature.

Computed values of α_1, α_2 and α_3 are listed in Table 1 at various values of Ω, for $Pr = 0.7$. Using these values neutral stability and amplification contours can be constructed. These are shown in Fig. 1.

RESULTS AND DISCUSSION

Equation (19) was integrated inwards, that is, towards the centerline of the plume from its outer edge, by making use of the asymptotic form for $S_1(\eta)$ as $\eta \to \infty$. A similar procedure was used in obtaining α_2 and α_3 from equations (23) and (24), respectively. The neutral stability curve, i.e. $\Omega(G)$ on which $\alpha_i \equiv 0$, is obtained by solving

$$\alpha_i = \alpha_{1,i} + \varepsilon_M \alpha_{2,i} + \varepsilon_H \alpha_{3,i} \equiv 0. \qquad (27)$$

The value of G at a given Ω that one obtains from equation (27) depends on the value of $\bar{R}$. In all the computations here, $\bar{R}$ has been taken to be unity. The neutral curve so obtained is shown in Fig. 1 along with contours of constant amplification. These latter curves represent the exponential growth of a disturbance of fixed frequency, i.e. ω, as it crosses the neutral curve and propagates downstream. If A_n is the amplitude of a disturbance at a downstream location corresponding to neutral stability and A_x is its amplitude further downstream, then

$$A_x/A_n = e^A, \; A = -\int_{x_n}^{x} \alpha_i \, dx/\delta = -5/3 \left(\int_{G_n}^{G} \alpha_i \, dG \right) \qquad (28)$$

with α_i being the imaginary part of α. The neutral curve is $A \equiv 0$. Curves of constant amplification have been obtained by determining α_i at various values of G, keeping ω fixed. The integral in equation (28) is then evaluated by the simple trapezoidal rule, with a step size in G of 2.5. Also shown for comparison, in Fig. 1, are the neutral curve and contours of constant amplification for a natural convection plume.

It is clear from Fig. 1 by comparing the neutral curves and those for $A = 2$, that the mixed convection effect stabilizes the flow considerably. The reason for this enhanced stability, lies in the changed nature of the velocity profile for the base flow, near the inflection point. Details of this line of reasoning can be found in ref. [9].

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APPENDIX. GOVERNING EQUATIONS OF THE BASE FLOW

At $O(\varepsilon_M)$:

$$F_2''' + 1/5(3F_1F_2'' + 2F_2F_1'' - F_1'F_2') + H_2 = 0$$

$$H_2'' + 1/5Pr(3F_1H_2' + 2F_2H_1' + 4H_2F_1' + 3H_1F_2') = 0$$

$$F_2(0) = F_2''(0) = F_2'(\infty) - 1 = H_2'(0) = H_2(\infty) = 0$$

$$F_2'(0) = 0.05982, \quad H_2(0) = -0.21831.$$